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APPLICATIONS OF FUNCTIONS OF THE COMPLEX VARIABLE TO CERTAIN ISOTHERMIC SYSTEMS

BY

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Thesis for the Degree of Bachelor of Arts
in Mathematics

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UNIVERSITY OF ILLINOIS

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Introduction.

1. Statement of problem. It has been our purpose to collect specific problems showing the application of the complex variable to physics, particularly to the two branches of physics, heat and activity. The development of the theory of the complex variable is comparatively recent, so that it has not yet been widely applied to physics. The complex variable $x+iy$ represents a point having direction; in general, then, its application is to problems involving direction. We shall discuss

how the functions of the complex variable may be applied in determining the character of lines of force and equipotential lines of different kinds of motion in two dimensional space. To this end we have made a study of 1, isogonal transformations, the mapping of various isothermic systems into one another; 2) how these systems represent the lines of force and of potential of different kinds of motion; 3) and, to a certain extent, we have worked out experimentally in the physics laboratory and obtained results which show that what we have worked out mathematically, holds good in the physical world.

Before we take up the first chapter, a

few words are necessary in explanation of what we mean by 1.) an analytic function of a complex variable $z = x+iy$; 2.) curvilinear co-ordinates; 3.) the u and v curves, or lines of flow and equipotential lines; 4.) the definition of isothermic systems.

2. The analytic function. We define an analytic function according to Riemann. In order that w be an analytic function of the complex variable z, where

$$z = x+iy,$$

such a function w, must have for every assigned value of the variable not only a definite value or system of values, but also for each of these values,

a definite differential coefficient. The advantage of this definition is, that it is quite independent of the existence of an algebraic expression for the function. The question now arises, what are the conditions under which w may be an analytic function of z.

Given that

$$w = u + i\,v = f(z) = f(x + iy),$$

where u and v are real variables. Taking the partial derivative of w with respect to y we have

$$\frac{\partial w}{\partial y} = \frac{\partial (u + i\,v)}{\partial y} = i\,\frac{\partial (u + i\,v)}{\partial x},$$

or

$$\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} = i\,\frac{\partial u}{\partial x} + i\,\frac{\partial v \cdot i}{\partial x} = i\,\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}$$

Equating real and imaginary parts,

we obtain

(1.) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

(2.) $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

It follows, then, that (1.) and (2.) are the equations of condition, which the real and imaginary parts of a complex function must satisfy in order that it be a function of a complex variable. If u and v satisfy (1.) and (2.) then $u + iv$ is a function of the complex variable z.

The partial differentiation of (1.) with respect to x gives

(3.) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \frac{\partial v}{\partial y}}{\partial x}$$

Likewise the partial differentiation of (2.) with respect to y, gives

(4.) $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial \frac{\partial v}{\partial x}}{\partial y} = -\frac{\partial \frac{\partial v}{\partial y}}{\partial x}$$

From (3.) and (4.), we have

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{5}$$

Differentiating (1.) with respect to y, and (2.) with respect to x, we obtain also

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0 \tag{6}$$

If $u + iv$ is a function of the complex variable $x + iy$, the real part u and the imaginary part v satisfy the partial differential equations (5.) and (6.). Equations (5.) and (6.) are the same as Laplace's equation for the potential function in two dimensional space. They are usually written in the abbreviated form

$$\Delta u = 0$$

3. Curvilinear co-ordinates. Let us assume that

$$\Phi(x,y) = \text{const.}\ ,\quad \Psi(x,y) = \text{const.}$$

are two systems of curves in a plane. If we choose any definite point (x,y), then Φ and Ψ will have certain values u and v; that is

$$\Phi(x,y) = u\ ,\quad \Psi(x,y) = v$$

To each point (x,y) corresponds, then, a pair of values (u,v). We can also write

$$\varphi(u,v) = x\ ,\quad \psi(u,v) = y$$

To each point (u,v) corresponds also a point (x,y).

We can use u and v instead of x and y as the determining co-ordinates of the point. By means of rectangular co-ordinates, we locate a point as the intersection of a certain line, parallel to the y-axis, with a certain other line

parallel to the x-axis. If we employ u and v instead of x and y, the point in question will be on the curve

$$\Phi(x, y) = u$$

and also on the curve

$$\Psi(x, y) = v;$$

that is, we determine the point as the intersection of two curves. u and v are called the curvilinear co-ordinates or parameters of the point. The curves giving at this formulation are called the parameter lines.

In the ordinary rectangular co-ordinate system the parameter lines are two systems of parallel lines perpendicular to each other. In oblique systems, they are any two systems of parallel

lines. In polar co-ordinates, the parameter lines are concentric circles and lines through the common center. In bicircular co-ordinates; that is, when we locate the point by its distances u and v from two fixed points, the poles, the parameter lines are circles about each pole, and the orthogonal system through the fixed points.

Examples of curvilinear co-ordinates are the location of a point on the earth's surface by longitude and latitude and the location of a star by right ascension and declination.

4. Definition of an isothermic system.

Through two orthogonal parallel systems, the z plane can be divided into a

systems of congruent rectangles, which in a special case may be squares. If a plane so divided is mapped, by means of a function of a complex variable

$$w = f(t),$$

on the w-plane, then the two parallel systems are transformed into two curve systems, which intersect at right angles because of the conformity of the mapping. The division of the w-plane is into small right-angled plane portions, and hence, if the rectangles of the z-plane are indefinitely small, we can say that the w-plane is subdivided into indefinitely small rectangles. A co-ordinate system through which the plane can be divided in-

to indefinitely small rectangles is called an isometric or isothermic system.

We shall now show that two systems of curves u and v, which satisfy the partial differential equation

$$\Delta u = 0,$$

represent an isothermic system. In other words, an isothermic system may be defined as one which satisfies Laplace's equation, or as one by which the plane can be divided into indefinitely small rectangles.

Given that

$$u + i v = f(x + i y),$$

let the inverse function be

(1.) $$x + i y = \phi(u + i v)$$

Forming the conjugate function we have

$$x - iy = \phi_1(u - iv) \tag{2.}$$

By addition of (1.) and (2.), we obtain

$$x = \frac{\phi(u+iv) + \phi_1(u-iv)}{2} \tag{3.}$$

By subtraction, we obtain

$$y = \frac{\phi(u+iv) - \phi_1(u-iv)}{2i} \tag{4.}$$

It follows that, corresponding to the parallels

$$x = a \quad , \quad y = b$$

of the z-plane, we have the orthogonal systems of curves,

$$\frac{\phi(u+iv) + \phi_1(u-iv)}{2} = a \tag{5.}$$

$$\frac{\phi(u+iv) - \phi_1(u-iv)}{2i} = b \tag{6.}$$

Since the division of the z-plane consists of a multiply congruent rectangles, if a and b take values in arithmetic progression, the divisions of the w-plane are in small rectangles. Let a and b

take the following values:

$$a = \ldots\ldots, -3\alpha, -2\alpha, -\alpha, 0, \alpha, 2\alpha, 3\alpha, \ldots\ldots$$

$$b = \ldots\ldots, -3\beta, -2\beta, -\beta, 0, \beta, 2\beta, 3\beta, \ldots\ldots$$

When $\alpha = \beta$ we have the division in small squares. The terms a and b are called the isothermic parameters of the two curve systems. x is the real part of the function of the complex variable $\phi(u+iv)$, and y is the imaginary part, consequently, they satisfy Laplace's equation. Hence, the left sides of equations (5) and (6) satisfy Laplace's equation. Therefore we may define an isothermic system as one which, when written in Cartesian co-ordinates, in the real form

$$u = c \quad ,$$

satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The name isothermic rests upon its physical meaning which will be explained in the next paragraph.

It is evident that by means of the function

$$u + i v = f(x + i y),$$

we might have passed from the w-plane to the z-plane, since the conjugate function gives

$$u - i v = f_1(x - i y)$$

Then it follows that

$$u = \frac{f(x+iy) + f_1(x-iy)}{2}$$

$$v = \frac{f(x+iy) - f_1(x-iy)}{2i},$$

and the curves

$$u = \text{const.}, \quad v = \text{const.}$$

correspond to the orthogonal systems; viz.

$$\frac{f(x+iy) + f_1(x-iy)}{2} = a$$

$$\frac{f(x+iy) - f_1(x-iy)}{2i} = b$$

The orthogonality of the two systems follows analytically from the differential equation

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$$

For if

$$u = f(x, y) = c$$

is the equation of a curve system, its differential gives

$$du = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} \frac{dy}{dx} = 0$$

Then it follows that

$$\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = -\frac{dy}{dx}$$

In order that two curves

$$f(x,y) = 0 \quad , \quad f(x_1, y_1) = 0$$

be orthogonal the f must satisfy the condition that

$$\frac{dy}{dx} = -\frac{dx_1}{dy_1}$$

It follows, then, that the two systems of curves $f(x,y)$ and $f_1(x,y)$ are orthogonal if

$$\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = -\frac{\frac{\partial f_1(x,y)}{\partial y}}{\frac{\partial f_1(x,y)}{\partial x}}$$

Substituting u and v for $f(x,y)$ and $f_1(x,y)$ respectively and clearing of fractions we have

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} = 0$$

This is the form of the differential equations (1.) and (2.) §2. Hence, the condition that

$$u = \text{const.}, \quad v = \text{c. s.}$$

be orthogonal is that these partial

differential equations exist.

5. Physical interpretation of the u and v curves. A line at every point of which the potential has the same value and which is therefore called a equipotential line, is such, that the attraction is everywhere in the direction of its normal. Similarly an isotherm is a line at every point of which the temperature is the same, and the flow of heat is in the direction of its normal. A line drawn from any center, so that at every point of its length its tangent is the direction of the attraction at that point, is called a line of force or of flow. A line of flow obviously cuts at right angles every

equipotential line or isothermic which it meets. If we take u as the function of flow, v as the potential function then the curves

$$u = \text{const.}, \quad v = \text{const.}$$

are respectively the lines of flow and of equipotential. Since the lines of flow intersect the equipotential lines at right angles, it follows, in accordance with the definition of an isothermic system (§ 7), that the curves

$$u = \text{const.}, \quad v = \text{const.}$$

represent an isothermic system. Hence the functions u and v satisfy Laplace's equation, which is the same condition that

$$u + i v = f(x + iy)$$

Hence in any assumption that

$$u + iv = f(x+iy)$$

the curves

$$u = \text{const.}$$

may represent the lines of flow; the curves

$$v = \text{const}$$

may represent the lines of equipotential or vice versa.

Chapter I

Isothermic Systems

1. Conformal representation. If we represent w and z in two different planes, then $w = f(z)$ represents a mapping of the z-plane on the w-plane. The process of mapping the one plane upon the other consists in separating $w = f(z)$ into its real and imaginary parts. viz:

$$w = u + iv = f(z) = f(x+iy) = \varphi(x,y) + i\psi(x,y)$$

Then by equating the real and imaginary parts separately, that is placing

$$u = \varphi(x,y), \quad v = \psi(x,y),$$

we obtain a relation between the u, v

co-ordinates of the w-plane and the x, y co-ordinates of the z-plane. This relation gives us the substitution which will transform any curve of the one plane to the corresponding curve of the other plane. This process is called mapping.

It is shown in the theory of analytic functions, that when w is an analytic function of z the mapping is conformal; that is, that two curves of the one plane intersect in the same angle when mapped on the other plane. Such a mapping of one plane upon another is called conformal representation, which is the English translation of the German "Konforme

Abbildung". Because of the conformity of the representation, an isothermic system on the z plane is transformed into another isothermic system on the w-plane. Hence, by means of the analytic function, we are enabled to get the analytic expression and geometric representation of a variety of isothermic systems.

2. The simplest isothermic system. When we have given the two orthogonal systems of parallel lines

$$x = \text{const.}$$

$$y = \text{const.}$$

we have the simplest isothermic system. Regarding x and y as real variables, we may consider them as the Cartesian

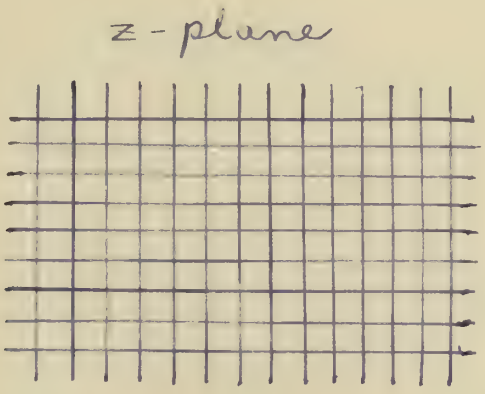
co-ordinates, and have

(1) $x = a$

(2) $y = b$

The left sides satisfy Laplacis equation and it is evident that the configuration is in small rectangles. Hence the curves (1.) and (2.) represent an isothermic system. Fig. 1. is the representation of this system.

Fig. 1.



3. Mapping of the function $w = e^z$. Given

$$w = e^z$$

where

$$w = u + iv \quad , \quad z = x + iy$$

Let us consider the mapping of the z plane upon the w-plane in accordance with the principles set forth in §1. Then we may write

$$u + iv = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y),$$

since

$$e^{iy} = \cos y + i \sin y.$$

Equating real and imaginary parts, we have

(1) $$u = e^x \cos y$$

(2) $$v = e^x \sin y$$

Laplace's equation is satisfied; for

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x \sin y - e^x \sin y = 0$$

Hence the mapping is conformal.

Dividing (2) by (1), we obtain

$$\frac{v}{u} = \frac{\sin y}{\cos y} = \tan y$$

or

(3.) $$\text{arc}\tan\frac{v}{u} = y = \varphi$$

Squaring (1.) and (2.) and dividing by e^x, we have

$$\frac{u^2}{e^{2x}} = \cos^2 y$$

$$\frac{v^2}{e^{2x}} = \sin^2 y$$

By addition we obtain

$$\frac{u^2+v^2}{e^{2x}} = \sin^2 y + \cos^2 y = 1.$$

or

$$u^2+v^2 = e^{2x}$$

(4.) $$\log\sqrt{u^2+v^2} = \log\rho = x$$

Equations (3.) and (4.) give us ... relation between the u, v co-ordinates of the w ... plane and the corresponding x, y co-ordinates of the z-plane, which is necessary to map any configuration

from the one plane on to the other plane; that is, they give us the relation between the two co-ordinate systems.

The equation ... w axis which curves of the w-plane correspond to the lines

$$x = \text{const.}, \quad y = \text{const.}$$

on the z-plane. Substitute in (3.) and (4.)

$$x = a \quad , \quad y = b.$$

It follows that

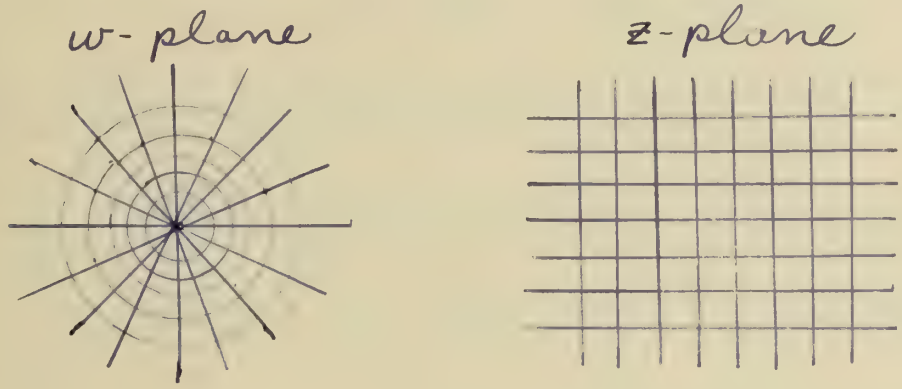
(5.) $$\arctan \frac{v}{u} = \varphi = b$$

(6.) $$\log \sqrt{u^2 + v^2} = \log r = a$$

Equation (5.) represents a pencil of lines through the origin. (6.) is the equation of the concentric circles about the origin as a center. Our new isothermic system is, then, made up by the bundle of rays through

the origin, and its orthogonal system, the concentric circles. The following figure shows this mapping.

Fig. 2.



$$w = e^{z}$$

In other words, corresponding to the isothermic system

$$x = a \quad , \quad y = b$$

of the z-plane, we have in the w-plane the isothermic system consisting of the pencil of rays through the origin and the orthogonal system of concentric circles about the

origin.

The inverse function, $w = \log z$, as is evident, would map the curves

$$u = \text{const.}, \quad v = \text{const.}$$

into the concentric circles and its orthogonal system of rays through the origin. This may be shown independently in the same manner as employed above.

4. Mapping of $w = \frac{1}{z}$. Separating the function into its real and imaginary parts, we have

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Equating the real and the imaginary parts we obtain

$$(A) \qquad u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}$$

Equations (A) satisfy Laplace's equation and therefore the mapping is

conformal. Equations (A) are the equations of transformation, which will change the equation of any curve in the w-plane to the equation of the corresponding curve in the z-plane. Solving these for x and y in terms of u and v we have the relations which would give us the inverse transformation from the z to the w-plane.

Expressing the given relation of w and z in terms of polar co-ordinates, we may write

$$\rho(\cos\Phi + i\sin\Phi) = \frac{1}{r(\cos\theta + i\sin\theta} = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right]$$

where

$$w = \rho(\cos\Phi + i\sin\Phi), \quad z = r(\cos\theta + i\sin\theta)$$

That is, by the mapping $w = \frac{1}{z}$ we have a reflection with respect to the

u-axis of reals plus an inversion of the z-plane. In the two planes we have the corresponding curves:

$$\begin{cases} (1.)\ f(\rho, \phi) = 0 \\ (2.)\ f(u, v) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1.')\ f\left(\frac{1}{r}, -\theta\right) = 0 \\ (2')\ f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = 0 \end{cases}$$

We shall now consider what is the mapping in the z-plane of the circles

$$\rho = \text{const.}$$

about any point $a+ib$ of the w-plane, and of the system of rays

$$\phi = \text{const.}$$

The equation of the circle about the point $a+ib$ of the w-plane as a center is,

$$(3.) \qquad (u-a)^2 + (v-b)^2 = c^2$$

or

(4) $$[u+iv-(a+ib)]\cdot[u-iv-(a-ib)]=c^2$$

Substituting in equation (4) the values

$$u+iv=\frac{1}{x+iy}$$

$$u-iv=\frac{1}{x-iy}$$

we obtain

$$\left[\frac{1}{x+iy}-(a+ib)\right]\cdot\left[\frac{1}{x+iy}-(a-ib)\right]=c^2$$

If we take $\frac{a+ib}{x+iy}$ out of the bracket and substitute $\frac{a-ib}{a^2+b^2}$ for its equivalent $\frac{1}{a+ib}$, and $\frac{a+ib}{a^2+b^2}$ for its equivalent $\frac{1}{a-ib}$, we have

$$\frac{a^2+b^2}{x^2+y^2}\left[x-\frac{a}{a^2+b^2}+i\left(y+\frac{b}{a^2+b^2}\right)\right]\cdot\left[x-\frac{a}{a^2+b^2}-i\left(y+\frac{b}{a^2+b^2}\right)\right]=c^2,$$

or finally

(5.)

$$\frac{a^2+b^2}{x^2+y^2}\left[\left(x-\frac{a}{a^2+b^2}\right)^2+\left(y+\frac{b}{a^2+b^2}\right)^2\right]=c^2$$

The function in the bracket, taken absolutely, is the square of the radius vector, r, proceeding from the point

$\frac{a-ib}{a^2+b^2}$ to the variable point $x+iy$. Since

$$\frac{a-ib}{a^2+b^2} = \frac{1}{a+ib}$$

the radius vector, r_1, is measured from the point $\frac{1}{a+ib}$. If now we substitute

$$x^2+y^2 = r^2$$

in (5.) and extract the root, the equation of our curve becomes

(6.) $$\frac{r_1}{r}\sqrt{a^2+b^2} = c$$

The curves,* given by a constant ratio between two radii vectors, represent a system of circles about the points from which the radii vectors emanate. Equation (6) then, represents a system of circles about the point i and about the

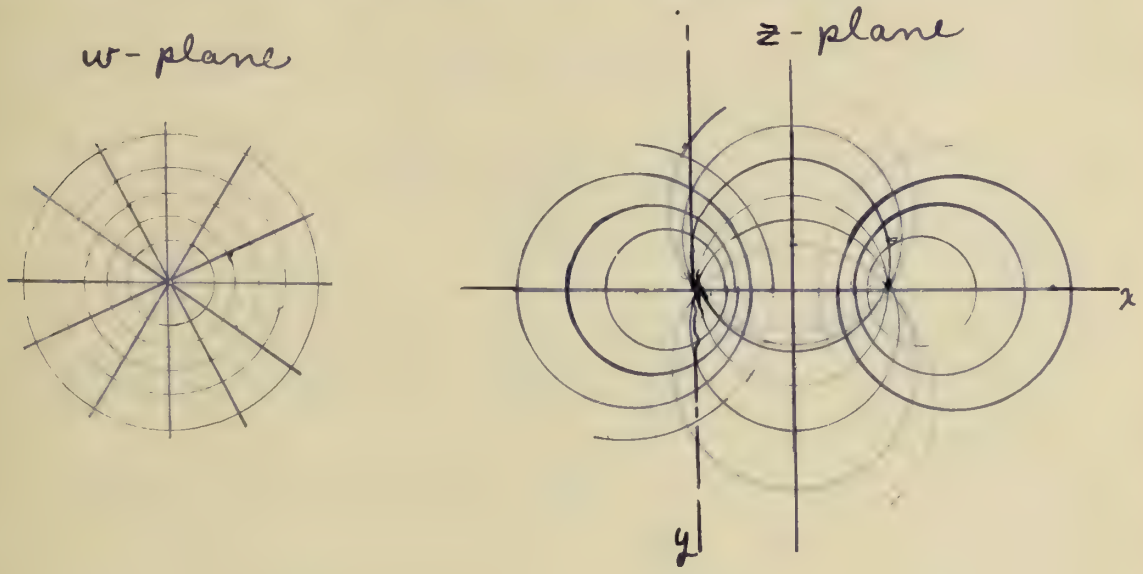
* This mapping is fully discussed by F. M. Cole in the annals of mathematics v. 5. No 4.

the point $a+ib$. The mapping of the curves

$$\rho = c, \quad \varphi = \chi$$

is shown in the following figure.

Fig. 3.



$$w = \frac{1}{z}$$

We shall now consider the mapping of the curves

$$\varphi = \chi$$

The equation of the lines $\varphi = \chi$ through the point $a+ib$ is

$$\frac{v-b}{u-a} = \tan \chi, \text{ or } \arctan \frac{v-b}{u-a} = \chi$$

or finally

$$\frac{1}{2i}\log\frac{(u-a)+i(v-b)}{(u-a)-i(v-b)} = \frac{1}{2i}\log\frac{(u+iv)-(a+ib)}{(u-iv)-(a-ib)} = y$$

Substituting

$$u+iv = \frac{1}{x+iy}, \quad u-iv = \frac{1}{x-iy},$$

in the above equation we obtain

$$\frac{1}{2i}\log\frac{\frac{1}{x+iy}-(a+ib)}{\frac{1}{x-iy}-(a-ib)} = y$$

or

$$\frac{1}{2i}\log\frac{a+ib}{a-ib}\cdot\frac{x-iy}{x+iy}\cdot\frac{\frac{1}{a+ib}-(x+iy)}{\frac{1}{a-ib}-(x-iy)} = y$$

If we now substitute

$$\frac{1}{a+ib} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$$

$$\frac{1}{a-ib} = \frac{a}{a^2+b^2} + \frac{ib}{a^2+b^2}$$

we obtain

$$\frac{1}{2i}\log\frac{x+iy-\left(\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}\right)}{x-iy-\left(\frac{a}{a^2+b^2}+i\frac{b}{a^2+b^2}\right)} - \frac{1}{2i}\log\frac{x+iy}{x-iy} + \frac{1}{2i}\log\frac{a+ib}{a-ib} = y$$

If we write this in terms of the arc tangent, we have

$$(7) \qquad \arctan\frac{y+\frac{b}{a^2+b^2}}{x-\frac{a}{a^2+b^2}} - \arctan\frac{y}{x} + \arctan\frac{b}{a} = y$$

On the left side of this equation, we have the directing angles of the radii vectors. The first term stands for the directing angle of the vector r_1, emanating from the point $\frac{1}{a+ib}$, since;

$$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} = \frac{1}{a+ib}$$

and drawn to the point $x+iy$. The second gives the directing angle of the vector r, and the third, the directing angle of the vector from the the origin to $a+ib$. Hence we may write equation (7.) in the form

$$\theta_1 - \theta + \alpha = \gamma \tag{8.}$$

This is the equation of a pencil of circles through two points. In this case the two points are $\frac{1}{a+ib}$ and 0.

The mapping of the curves

$$\varphi = \gamma$$

is shown in Fig. 3.

The following theorem states the analytic relations, just derived, between the two planes.

Since the circles $\rho = c$ about the point $a + ib$ of the w-plane correspond to the system of circles $\frac{r_1}{r}\sqrt{a^2+b^2} = c$ of the z-plane, and the lines $\varphi = \gamma$ through this point correspond to the system of circles $\theta, -\theta + \alpha = \gamma$, then in the two planes there correspond the curves

$$f(\rho, \varphi) = 0 \quad \text{and} \quad f\left(\frac{r_1}{r}\sqrt{a^2+b^2}, (\theta, -\theta+\alpha)\right) = 0$$

We may consider the system of rays through a point $a + ib$ and its orthogonal system as a special case of the pencil of circles

$$\Phi - X = \chi$$

through two points $a+ib$ and $a,+ib,$

and its orthogonal system

$$\frac{\rho}{\alpha} = c \;;$$

where the point $a,+ib,$ is at ∞ and

in consequence

$$X = 0$$

By means of the function $w = \frac{1}{z}$, we transformed the pencil of circles through the points $a+ib$ and ∞ into another pencil of circles through the points $\frac{1}{a+ib}$ and $\frac{1}{\infty} = 0$. It follows that the pencil of circles

$$(\varphi - \theta + \alpha) - (\chi - \theta + \alpha_1) = \chi$$

or

$$(\varphi - \chi) = \gamma - (\alpha - \alpha_1)$$

through the points $\frac{1}{a+ib}$ and $\frac{1}{a,+ib,}$ of the z-plane corresponds to the pencil

of circles

$$\phi - \chi = \gamma$$

through the points $a+ib$ and a_1+ib_1 of the w-plane. Also the system

$$\frac{\frac{\rho}{r}\sqrt{a^2+b^2}}{\frac{\eta}{r}\sqrt{a_1^2+b_1^2}} = c$$

or

$$\frac{\rho}{\eta}\sqrt{\frac{a^2+b^2}{a_1^2+b_1^2}} = c$$

corresponds to the system

$$\frac{\rho}{\eta} = c$$

which is orthogonal to the pencil of circles.

It is also evident from the above that the pencil of circles

$$\phi - \chi = \gamma$$

and its orthogonal system is or may become, by a proper choice of co-ordinates,

the reciprocal of the systems

$$\varphi = y \quad , \quad r = c$$

5. *Mapping* $w = z^2$: The given function may be written in the form:

$$w = u + iv = x + iy = x^2 - y^2 + 2ixy$$

Equating real and imaginary parts we get

(1) $$u = x^2 - y^2$$

(2.) $$v = 2xy$$

From these equations, we see that u, v both satisfy Laplace's equation; that is to say, we have

$$\Delta u = 0 \quad , \quad \Delta v = 0$$

Equations (1) and (2.) are the equations of transformation which will transform the equation of any curve in the w-plane to the equation of the

corresponding curve of the z-plane. By solving these for x, y in terms of u, v we should have the inverse transformations.

In polar co-ordinates, we have

$$\rho(\cos\phi + i\sin\phi) = [r(\cos\theta + i\sin\theta)]^2$$
$$= r^2(\cos 2\theta + i\sin 2\theta)$$

Hence

$$\rho = r^2 \quad , \quad \phi = 2\theta .$$

In the w-plane and z-plane we have, then, the following corresponding systems, viz:

$$\begin{cases} f(u,v) = 0 \\ f(\rho,\phi) = 0 \end{cases} \quad \text{and} \quad \begin{cases} f[(x^2-y^2), 2xy] = 0 \\ f(r^2, 2\theta) = 0 \end{cases}$$

Hence to the curves

$$u = a \quad , \quad v = b$$

of the w-plane there correspond in the

z-plane the curves

$$x^2 - y^2 = a$$

$$2xy = b$$

These are the equations of two systems of equilateral hyperbolas. Their middle point is at the origin. The parallels

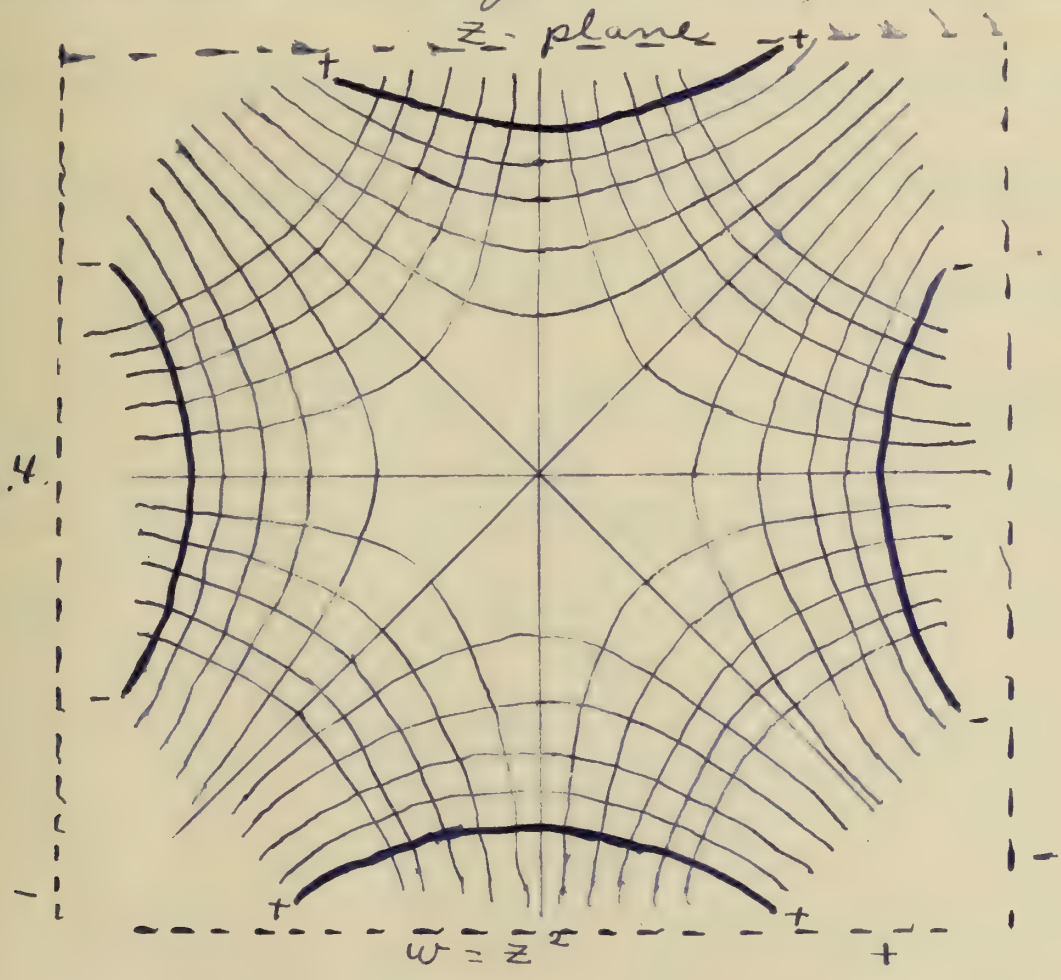


Fig. 4.

to the the u-axis correspond to the hyperbolas running asymptotic to the co-

ordinate axes. The parallels to the v-axis correspond to those running asymptotic to the lines bisecting the angles between the co-ordinate axes. This mapping is shown in Fig. 4.

We next inquire to what curves of the w-plane do parallels to the co-ordinate axes of the z-plane correspond. To map the lines parallel to the axis of reals on the w-plane, substitute in (1), and (2) $y = c$ and eliminate x. This gives,

$$u + c^2 = \left(\frac{v}{2c}\right)^2 \tag{3}$$

or

$$u^2 + v^2 = (u + 2c^2)^2$$

This is the equation of the parabola whose semi-axis is the u-axis. The focus is at the origin and the diretrix

is the line

$$u + 2c^2 = 0.$$

Since x is essentially real, c^2 is positive, and hence the directrix lies on the negative side of the u-axis, and the parabolas, then, extend out to infinity on the right of the directrix.

If in (1) and (2) we substitute $x = c$ and eliminate y, we have

$$c^2 - u = \left(\frac{v}{2c}\right)^2 \tag{4}$$

or

$$u^2 + v^2 = (u - 2c^2)^2$$

In this case the directrix

$$(u - 2c^2)^2 = 0$$

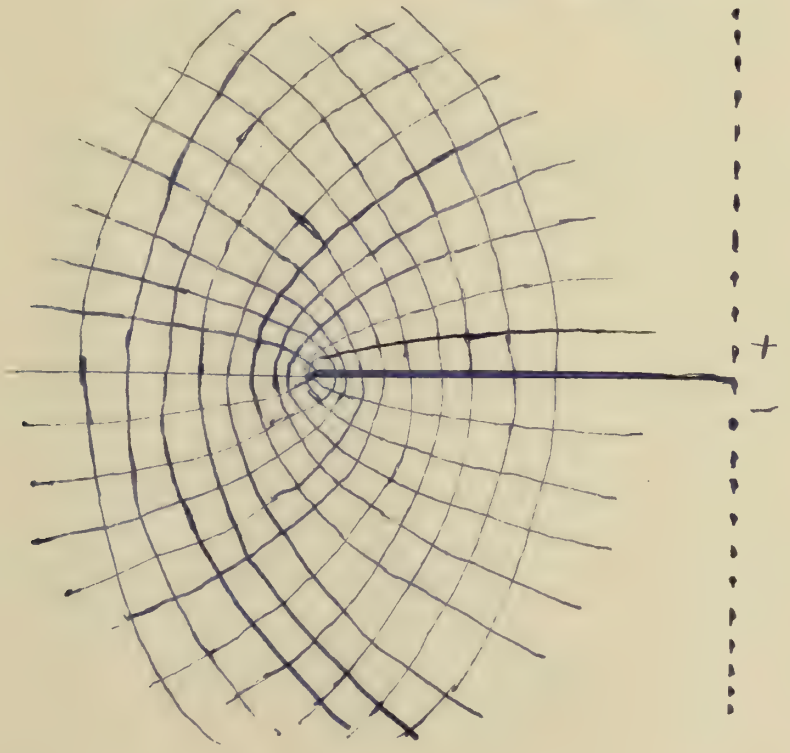
lies on the positive side of the u-axis and the parabolas (4) extend out to infinity on the left of the directrix.

By means of the transformation

$$w = z^2$$

the two orthogonal systems of parallel lines, $x = \text{const.}$, $y = \text{const.}$, are transformed into two orthogonal systems of parabolas, each system having the origin as focus and the imaginary axis as semi-axis. The mapping of the z-plane upon the w-plane is shown in the following figure.

Fig. 5



$w = z^2$

6. Mapping of the function $w = \sqrt{z}$. ~~The~~

The question to be answered is, to what curves of the z-plane do a pencil of rays through any point $u+iv$ of the w-plane and its orthogonal system of concentric circles correspond. The circle about the point $a+ib$ of the w-plane with radius c may be written

$$(u-a)^2 + (v-b)^2 = c^2$$

By expanding and factoring it follows that

1) $$[(u+iv)-(a+ib)]\cdot[(u-iv)-(a-ib)] = c^2$$

We may write the transformation

$$w = z^2$$

in the form

$$u+iv = (x+iy)^2,$$

Making this substitution in 1) we obtain

$$[(x+iy)^2-(a+ib)]\cdot[(x-iy)^2-(a+ib)]=c^2$$

Separating each bracket into its factors we get

$$[(x+iy)-\sqrt{a+ib}]\cdot[x+iy+\sqrt{a+ib}]\cdot[x-iy-\sqrt{a-ib}]\cdot[x-iy+\sqrt{a-ib}]=c^2$$

If now we put

$$\sqrt{a+ib}=\alpha+i\beta,$$

then

$$\sqrt{a-ib}=\alpha-i\beta,$$

and the equation of our curve becomes,

$$[x+iy-(\alpha+i\beta)]\cdot[x+iy+\alpha+i\beta]\cdot[x-iy-(\alpha-i\beta)]\cdot[x-iy+\alpha-i\beta]=c^2$$

Multiplying out the pairs of conjugate factors we have

$$[(x-\alpha)^2+(y-\beta)^2]\cdot[(x+\alpha)^2+(y+\beta)^2]=c^2$$

or

$$\rho_1^2\cdot\rho_2^2=c^2$$

(2.) $$\rho_1\cdot\rho_2=c,$$

where ρ_1 and ρ_2 are the radii vectores emanating from the points $\pm(\alpha+i\beta)$.

(2.) is the equation of a system of con-

focal lemniscates about the points, $\pm(\alpha+i\beta)$. Therefore by the mapping

$$w = \sqrt{z},$$

the circles $\rho = c$ about the point $a+ib$ correspond to confocal lemniscates about the points $\pm\sqrt{a+ib}$ in the z-plane.

The equation of a line which passes through the point $a+ib$ of the w-plane with the direction $\psi = \gamma$ can be written,

$$\arctan\frac{v-b}{x-a} = \frac{1}{2i}\log\frac{u+iv-(a+ib)}{u-iv-(a-ib} = \gamma$$

By the mapping $w = z^2$, it becomes

$$\frac{1}{2i}\log\frac{(x+iy)^2-a+ib}{(x-iy)^2-(a+ib)}$$

$$= \frac{1}{2i}\log\frac{[x+iy-(\alpha+i\beta)]}{[x-iy-(\alpha-i\beta)]}\cdot\frac{[x+iy+\alpha+i\beta]}{[x-iy+\alpha-i\beta]} = \gamma$$

or

$$= \frac{1}{2i}\log\frac{x+iy-(\alpha+i\beta)}{x-iy-(\alpha-i\beta)} + \frac{1}{2i}\log\frac{x+iy+\alpha+i\beta}{x-iy+\alpha-i\beta} = \gamma$$

$$= \arctan \frac{y-\beta}{x-\alpha} + \arctan \frac{y+\beta}{x+\alpha} = \gamma,$$

or finally

$$\theta_1 + \theta_2 = \gamma \tag{3.}$$

where θ_1 and θ_2 are the directions of the radii vectors ρ_1 and ρ_2. Equation (3.) is the equation of the equilateral hyperbola through the points $\pm\sqrt{a+ib}$. Hence the pencil of rays in the w-plane through the point $a+ib$ corresponds to the equilateral hyperbolas (3.) through the points $\pm\sqrt{a+ib}$.

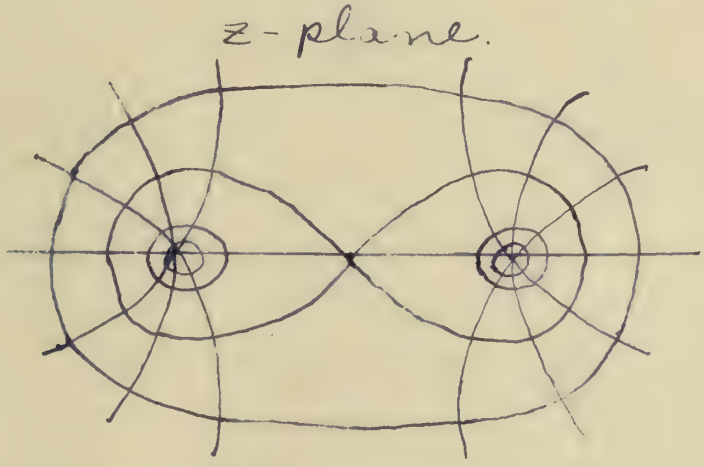
Equations (2.) and (3.) written in Cartesian co-ordinates in the real form were

$$\frac{1}{2}\left[\log\left\{(x-\alpha)^2+(y-\beta)^2\right\}+\log\left\{(x+\alpha)^2+(y+\beta)^2\right\}\right] = C$$

$$\arctan \frac{y-\beta}{x-\alpha} + \arctan \frac{y+\beta}{x+\alpha} = \gamma$$

The left hand members of the two equations satisfy Laplace's equation, hence the mapping is conformal. If c and y are given values in arithmetical progression, we have the rectangular subdivision. The mapping of this function is shown in the figure below.

Fig. 6



$w = \sqrt{z}$

7. Mapping of the function $w = \sqrt{z}$ concluded. Since, for the mapping $w = \sqrt{z}$, lemniscates correspond to all circles of the w-plane, then to a system

of circles through two points there corresponds a system of lemniscates through two pairs of points of the z-plane. The orthogonal system of circles becomes an orthogonal system of lemniscates. The equations of the two systems of circles are

$$\Phi - X = \gamma \tag{1.}$$

$$\log \frac{\rho}{2} = c, \tag{2.}$$

where $a_1 + ib_1$ and $a_2 + ib_2$ are the two points from which the radii vectores emanate. From equations (2.) and (3.) of §6, the corresponding systems of lemniscates are

$$(\Phi_1 + \Phi_2) - (X_1 + X_2) = \gamma \tag{3.}$$

$$\log \frac{\rho_1}{\varrho_1} \cdot \frac{\rho_2}{\varrho_2} = c, \tag{4.}$$

or simply

$$\frac{\rho_1}{\varrho_1} \cdot \frac{\rho_2}{\varrho_2} = C$$

The radii vectores $\rho_1, \rho_2, \varrho_1, \varrho_2$ issue from

the pairs of points $\pm\sqrt{a_1+ib_1}$ and $\pm\sqrt{a_2+ib_2}$, and ϕ_1, ϕ_2, χ_1, χ_2 are the corresponding directing angles.

If we put (3) and (4.) into Cartesian coordinates, where the choice of origin is optional as well as the direction of the real axis, then the left sides will satisfy Laplace's equation. Figures 7, 8, 9, represent the mapping of this function. The quadratic subdivision may be seen from the figures.

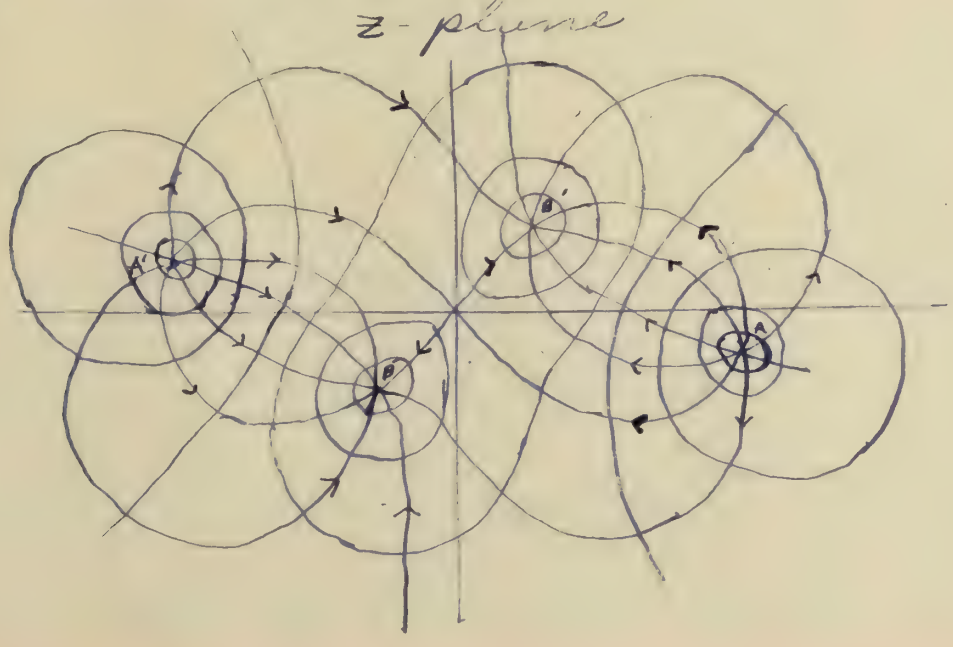


Fig. 7. is the most general case. Fig. 8. is a special case where

$$a_1 = 1 \quad , \quad b_1 = 0$$

$$a_2 = -1 \quad , \quad b_2 = 0.$$

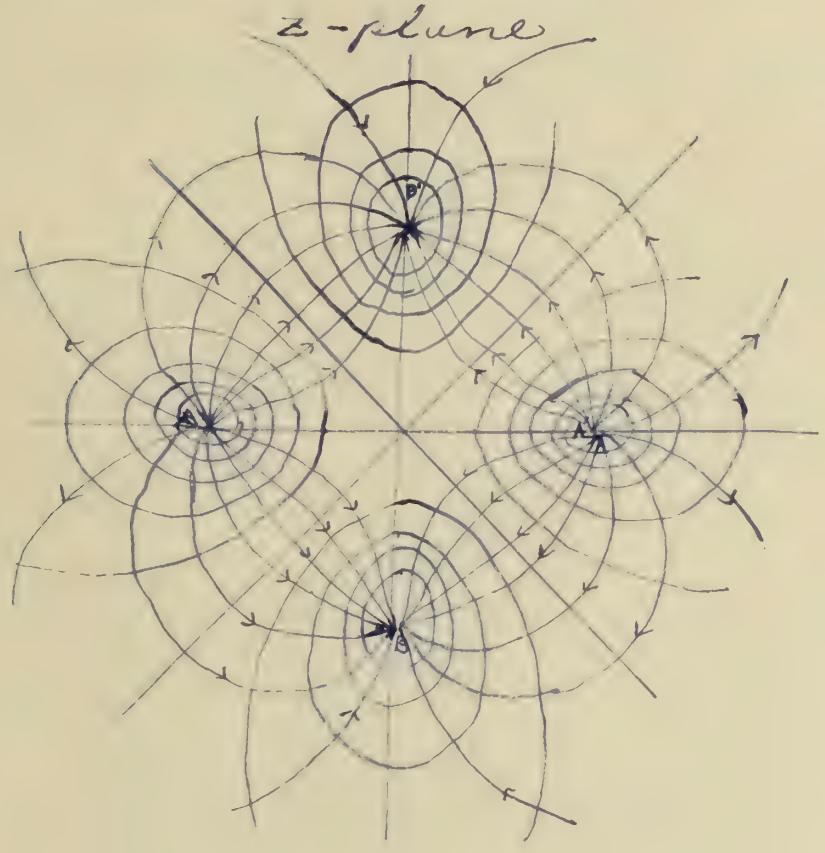


Fig. 8.

Fig. 9. is the case where $a_1 + ib_1$ is at the origin. Since a pencil of circles through two points, one of which is the origin, together with its orthog-

onal system is connected with a pencil of rays and its orthogonal system by means of the function $w = \frac{1}{z}$, it follows that Fig. 9 also represents the mapping of the curves

$$r = c \quad , \quad \varphi = \gamma .$$

of the z-plane upon the w-plane by means of the function

$$w = \frac{1}{\sqrt{z}} .$$

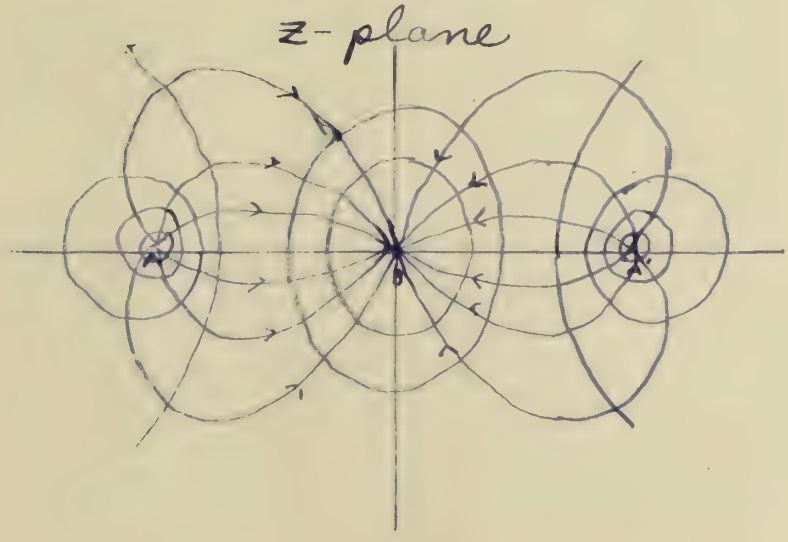


Fig. 9

8. *Mapping of* $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$. The given function may be written in the form

$$w = u + iv = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$= \frac{1}{2}\left[r(\cos\phi + i\sin\phi) + \frac{1}{r}(\cos\phi - i\sin\phi)\right]$$

$$= \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\phi + \frac{i}{2}\left(r - \frac{1}{r}\right)\sin\phi$$

Separating the real and imaginary parts, we obtain

(1.) $$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\phi$$

(2.) $$v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\phi$$

If now we eliminate ϕ by squaring (1.) and (2.) and then adding, we obtain

(3.) $$\frac{u^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r - \frac{1}{r}\right)} = 1$$

This is the equation of an ellipse whose semi-axes are $\frac{1}{2}\left(r + \frac{1}{r}\right)$ and $\frac{1}{2}\left(r - \frac{1}{r}\right)$ and whose foci are at the points ± 1. Since the ellipse is the locus of a point, the sum of whose distances from two fixed points, called the foci, is a

constant, and since this constant is equal to the major axis, we may write equation (3) in the form

$$P + Q = r + \frac{1}{r}, \tag{4.}$$

where P and Q are the distances of the point from the foci. It follows that the system of confocal ellipses

$$\frac{P+Q}{2} = \frac{1}{2}\left(c + \frac{1}{c}\right)$$

about the points ± 1 of the w-plane correspond to the system of circles

$$r = c$$

about the origin of the z-plane.

Also from (1.) and (2.) we get

$$\frac{u^2}{\cos^2\varphi} = \frac{1}{4}\left(r + \frac{1}{r}\right)^2$$

$$\frac{v^2}{\sin^2\varphi} = \frac{1}{4}\left(r - \frac{1}{r}\right)^2$$

Eliminating r, we have

(5.) $$\frac{u^2}{\cos^2\varphi} - \frac{v^2}{\sin^2\varphi} = 1$$

This is the equation of the hyperbola whose semi-axis is $\cos\varphi$ and whose foci are the points ± 1. (5.) may be written

(6.) $$P - Q = 2\cos\varphi$$

since the hyperbola is the locus of a point the difference of whose distances from two fixed points, viz: the foci, is constant. It follows, then, that the system of confocal hyperbolas

$$\frac{P-Q}{2} = \cos y, \text{ or } \arccos\frac{P-Q}{2} = y$$

about the points ± 1 of the w-plane correspond to the pencil of rays

$$\varphi = y$$

through the origin of the z-plane.

Equations (4.) and (5.) show the relations between the co-ordinates u, v of the

w-plane and the co-ordinates x, y of the z-plane.

This transformation is shown in Fig. 10. The two systems of curves represented by equations (4.) and (5.) are orthogonal because of the conformity of the transformation. The subdivision of the plane into small rectangles may be seen from the figure.

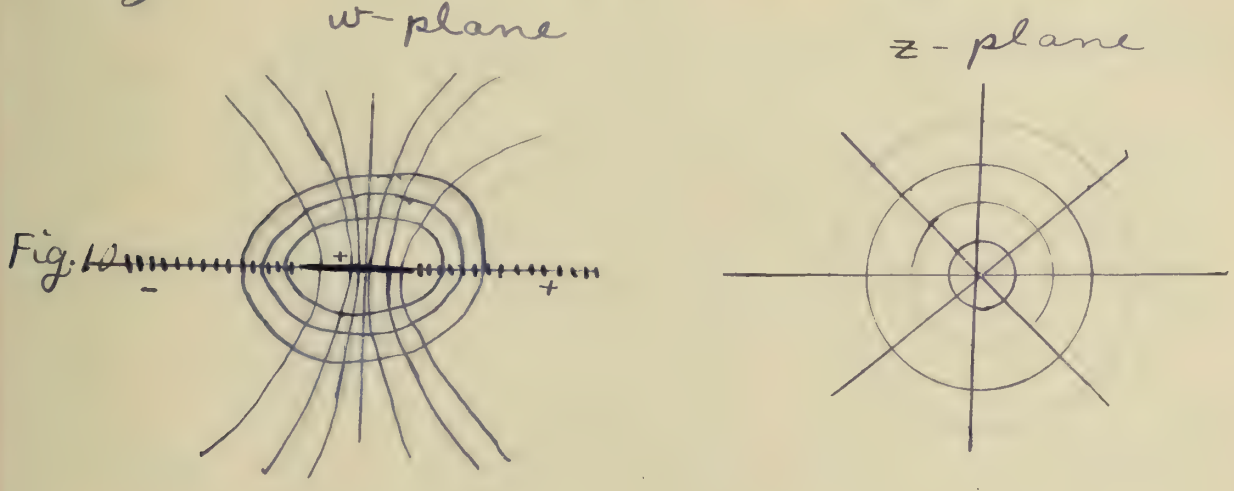


Fig. 10

9. Mapping of the function $w = z + \sqrt{z^2 - 1}$.

If in equations (4.) and (6.) §8, we solve for c and γ respectively, we find that

in the two planes these systems of curves correspond, viz:

$$f\left[\left(\frac{\rho+2}{2}+\sqrt{\left(\frac{\rho+2}{2}\right)^2-1}\right),\left(\arccos\frac{\rho-2}{2}\right)\right]=0$$

and

$$f(r,\phi)=0$$

The function

$$z=w+\sqrt{w^2-1}$$

is the inverse of

$$w=\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)$$

as may be seen by solving the latter for z. In the preceding paragraph, we found the mapping of the curves

$$r=c\quad,\quad \phi=\gamma$$

of the z-plane upon the w-plane by means of the function

$$w=\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)$$

In this paragraph, we shall discuss

the mapping of the curves

$$\rho = c \quad , \quad \psi = \gamma$$

of the w-plane upon the z-plane by means of the same function. It is evident from the above that this transformation is identical with that of the curves

$$r = c \quad , \quad \phi = \gamma$$

of the z-plane upon the w-plane by means of the function

$$w = z + \sqrt{z^2 - 1}$$

We have given that

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

then

$$u + i v = \frac{1}{2} \frac{(x+iy)^2 + 1}{x + iy},$$

and

$$u - i v = \frac{1}{2} \frac{(x-iy)^2 + 1}{x - iy}$$

If we subtract $a+ib$ from both members of the first and $a-ib$, from the second of these equations, we have

$$(1) \qquad u+iv-(a+ib) = \frac{(x+iy)^2+1}{2(x+iy)} - (a+ib)$$

$$(2) \qquad u-iv-(a-ib) = \frac{(x-iy)^2+1}{2(x-iy)} - (a-ib)$$

Separating the right hand members of (1.) and (2.) into their factors we obtain

$$u+iv-(a+ib) = \frac{[x+iy-(a+ib+\sqrt{(a+ib)^2-1})]\cdot[x+iy-(a+ib-\sqrt{(a+ib)^2-1})]}{2(x+iy)}$$

$$u-iv-(a-ib) = \frac{[x-iy-(a+ib+\sqrt{(a+ib)^2-1})]\cdot[x-iy-(a+ib-\sqrt{(a+ib)^2-1})]}{2(x-iy)}$$

If we substitute $\alpha_1+i\beta_1$ and $\alpha_2+i\beta_2$ for the two values $a+ib \pm \sqrt{(a+ib)^2-1}$, we get

$$(3) \qquad u+iv = \frac{[x+iy-(\alpha_1+i\beta_1)]\cdot[x+iy-(\alpha_2+i\beta_2)]}{2(x+iy)}$$

$$(4) \qquad u-iv = \frac{[x-iy-(\alpha_1-i\beta_1)]\cdot[(x-iy)-(\alpha_2-i\beta_2)]}{2(x-iy)}$$

By multiplying together (3) and (4) we have

$$(u-a)^2+(v-b)^2 = \frac{[(x-\alpha_1)^2+(y-\beta_1)^2]\cdot[(x-\alpha_2)^2+(y-\beta_2)^2]}{4(x^2+y^2)}$$

We may write this in the form

$$\rho = \frac{p \cdot q}{2r}, \tag{5}$$

where ρ is the distance of the point $u+iv$ from $a+ib$ while p, q, and r are the distances of the point $x+iy$ from $(\alpha_1+i\beta_1)$, $(\alpha_2+i\beta_2)$ and the origin respectively.

Dividing equation (3.) by (4.) taking the logarithm of each side and multiplying by $\frac{1}{2i}$, we obtain

$$\frac{1}{2i}\log\frac{u+iv-(a+ib)}{u-iv-(a-ib)} = \frac{1}{2i}\log\frac{x+iy-(\alpha_1+i\beta_1)}{x-iy-(\alpha_1-i\beta_1)}$$

$$+\frac{1}{2i}\log\frac{x+iy-(\alpha_2+i\beta_2)}{x-iy-(\alpha_2-i\beta_2)} - \frac{1}{2i}\log\frac{x+iy}{x-iy}$$

This may be written in the more simple form

$$\psi = \varphi + \chi - \theta, \tag{6}$$

where ψ, φ, χ and θ are the directing angles of the vectors ρ, p, q and r Hence the systems of curves

$$\frac{\rho \cdot \varphi}{2\pi} = c \quad , \quad \phi + \lambda - \theta = \gamma$$

of the z-plane correspond respectively to the systems

$$\rho = c \quad , \quad \psi = \gamma$$

of the w-plane. The mapping of this function is shown in the following figure.

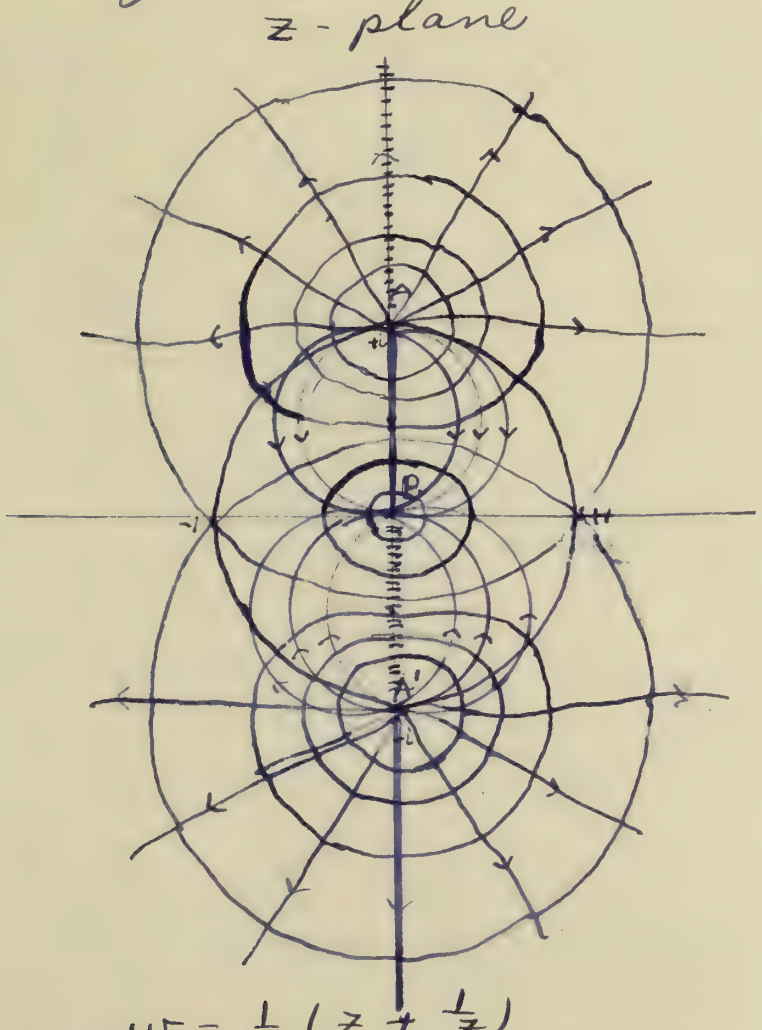


Fig. 11

Or calling this the w-plane, $w = z + \sqrt{z^2 - 1}$

If we put equations (5) and (6.) in a Cartesian co-ordinates, we obtain the form in which Laplace's equation is satisfied. This shows that the mapping is conformal.

10. Mapping of $w = \cos z$. The function

(1) $$w = \cos z$$

is analogous to the function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

The analogy may be seen by writing (1) in the expontial form

$$w = \cos z = \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right)$$ (1)

If we put

$$u + iv = \cos(x \pm iy),$$

(1) From Burkhardt's "Functionen Theorie", Vol. I: §40, eq. 6.

(2) From Burkhardt's "Functionen Theorie", Vol. I: §40, eq 8 and 9.

then we have

(2.) $u+iv = \cos x \cos iy - \sin x \sin iy$

(3.) $u-iv = \cos x \cos iy + \sin x \sin iy$

By adding (2.) and (3.), we obtain

(4.) $u = \cos x \cos iy$

Subtracting (3.) from (2.) we get

(5.) $iv = -\sin x \sin iy$

Substituting in (4.) and (5.)

$$\sin iy = \tfrac{1}{2}(e^{y}+e^{-y}),\quad \cos iy = \tfrac{1}{2}(e^{y}-e^{-y})^{*}$$

we obtain

$$u = \frac{e^{y}+e^{-y}}{2}\cos x,\quad v = -\frac{e^{y}-e^{-y}}{2}\sin x$$

By eliminating x and then y we have

(6.) $$\left(\frac{2u}{e^{y}+e^{-y}}\right)^2 + \left(\frac{2v}{e^{y}-e^{-y}}\right)^2 = 1$$

(7.) $$\left(\frac{u}{\cos x}\right)^2 + \left(\frac{v}{\sin x}\right)^2 = 1$$

* From Burkhardt's "Functionen theorie", Vol 1, § 40, ex. 8 and 9.

(6) is the equation of an ellipse. (7) is the equation of a hyperbola. By the mapping $w = \cos z$, the parallels to the axes of the z-planes correspond to confocal ellipses and hyperbolas of the w-plane, whose foci are the points ± 1.

The following figure shows the mapping of this function.

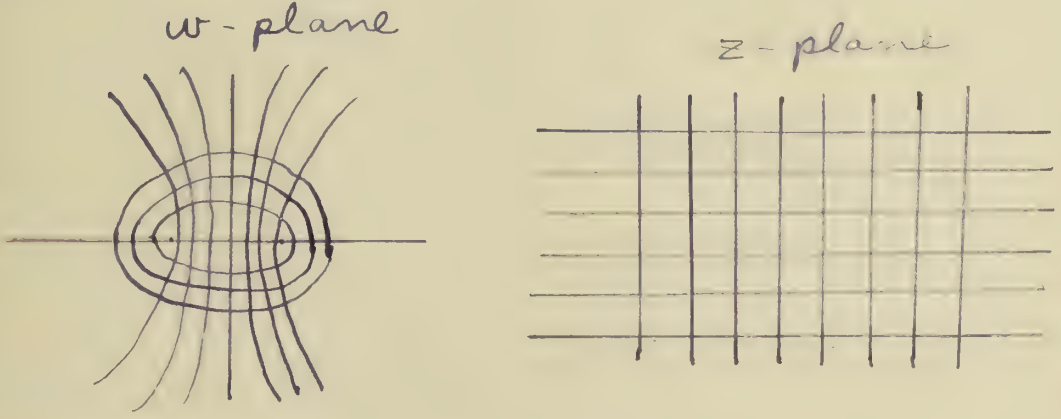


Fig. 12

$w = \cos z.$

Chapter II

Applications to Heat

1. Introduction. It is our purpose in this and the following chapters to show how the mappings just studied may be applied in physics. In the introduction, §5, we have already explained the physical meaning of the u and v curves. The question now arises what does conformal mapping mean in physical world and what is its use there. In general, we may state the application thus. By the mapping of one isothermic system into another

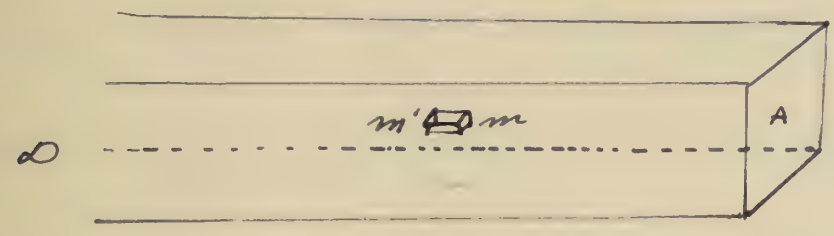
we obtain its conformal representation in another plane, and moreover we are enabled to determine the character of the lines of force and the equipotential lines of various problems in physics, without recourse to experiment. It will be evident that, since the complex variable represents a point in a plane and since the u and the v curves are therefore lines in a plane, we shall find applications in two-dimensional space only. In this chapter we shall consider certain applications to heat.

2. Laplace's equation in heat. However, before taking up any specific applications we shall show how Laplace's equa-

tion is arrived at and its significance in problems involving the potential function. Its significance is similar in the flow of heat, of electricity, and of liquids. Hence if we explain its meaning in heat, it will be sufficient.

Let v represent the temperature attained after a time t by an infinitesimal element of matter whose co-ordinates are x, y and z. Consider now the movement of heat in a prismatic element, enclosed between six planes perpendicular to the three axes of x, y, and z. The first three planes pass through the point m whose co-ordinates are x, y and z, and the others pass through the point m' whose

co-ordinates are $x+dx$, $y+dy$ and $z+dz$.



We suppose that at O is a constant source of heat and that the temperature surrounding the prismatic bar is maintained at 0°.

To find what quantity of heat enters the element during unit of time across the first plane passing through the point m and perpendicular to x, we must consider that the extent of the surface of the element on this plane is $dy\,dz$. The flow across this area is $-K\frac{dv}{dx}$*. K being the specific

* "The Analytical Theory of Heat" J. Fourier. Chapter I; §§ 92, 98

conductibility. Therefore across the rectangle $dy\,dz$, the element receives the quantity of heat $-K\,dy\,dz\,\frac{dv}{dx}$. The quantity which escapes from the element at m' is obtained by substituting $x+dx$ for x. Hence, the element loses at its second face, perpendicular to x, a quantity of heat equal to

$$-K\,dy\,dz\,\frac{dv}{dx} - K\,dy\,dz\,d\left(\frac{dv}{dx}\right)$$

The quantity of heat which remains in the element, then is equal to

$$-K\,dy\,dz\,\frac{dv}{dx} - \left(-K\,dy\,dz\,\frac{dv}{dx} - K\,dy\,dz\,d\,\frac{dv}{dx}\right)$$

$$= K\,dx\,dy\,dz\,\frac{d^2v}{dx^2}$$

This is the quantity of heat accumulated in the element in consequence of the flow in the direction of x. This accumulated heat would make the

temperature of the element vary, if it were not balanced by that which is lost in some other direction.

It is found in the same way that

$$K\,dx\,dy\,dz\,\frac{d^2v}{dy^2}$$

expresses the quantity of heat which the element acquires, in consequence of the flow in direction of y.

Also in same manner

$$K\,dx\,dy\,dz\,\frac{d^2v}{dz^2}$$

expresses the quantity of heat acquired by the element in consequence of the flow of heat in the direction of z.

In order that there be no change of temperature, it is necessary for the infinitesimal element to retain as much heat as it contained at first,

so that the heat it acquires in one direction must balance that which it loses in another. In other words the sum of the three quantities of heat acquired must be zero. Hence we must have.

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0 \quad *$$

For two-dimensional space, this equation becomes

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$$

3. Heat applied in an infinite line to a plane strip. This may be considered the simplest problem. Given a portion of a surface of infinite length,

* This is fully discussed by Fourier in Chap. I, Sec. IV, Pg 97

bounded by the lines

$$x_0 = \alpha_0 \quad , \quad x_1 = \alpha_1$$

we let the first constant be maintained at the temperature $0°$, the other, at $1°$. This plane strip is to be considered as a cross section of a prism of infinite height; for the reason that the loss of heat through radiation into the air would prevent a constant flow through the plane strip or surface. After a time a stationary condition is reached. The lines

$$x = \alpha$$

are the isotherms. The lines of flow of heat or the stream lines are

$$y = \text{const.}$$

The isotherms

$$x = \alpha$$

has the temperature

$$t = \frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0},$$

or, if we substitute for α the variable x,

$$t = \frac{x - \alpha_0}{\alpha_1 - \alpha_0}$$

This function satisfies Laplace's equation, within the strip is finite and single valued. It is evident that the stream lines

$$y = c$$

and the isothermes

$$x = \alpha$$

may be interchanged.

4. Heat applied to an infinite plane in one point. Instead of two parallel lines, we shall now consider two con-

centric circles of radii

$$\log r_0 = \alpha_0 \ , \ \log r_1 = \alpha_1$$

to be given the temperature 0° and 1° respectively. The isothermes are now the concentric circles

$$\log r = \alpha$$

and the temperature of the isothermes is given by the equation

$$t = \frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0} = \frac{\log \frac{r}{r_0}}{\log \frac{r_1}{r_0}} = \frac{\log \frac{\sqrt{(x-a)^2 + (y-b)^2}}{r_0}}{\log \frac{r_1}{r_0}} \qquad (1)$$

The right hand member which is written in Cartesian co-ordinates satisfies the partial differential equation

$$\Delta u = 0$$

Since t satisfies Laplace's equation it can be taken as the u function of a complex variable. The v part is deter-

mined except as to constant*: it is the orthogonal system which represents the lines of flow. This system is the pencil of rays

$$\theta = \arctan \frac{y}{x} \tag{2.}$$

where θ may be taken as the σ part of a complex variable, since (2.) satisfies

$$\Delta u = 0$$

We might, as will be evident consider the origin as our source of heat, which is radiated out into all directions along the lines

$$\theta = \arctan \frac{y}{x}$$

As in §3 we might interchange the lines of flow and the isothermes. The

* Burkhart § 33, III, IV.

The isotherms, then, would be the lines

$$\theta = \arctan \frac{y}{x}$$

The temperature would be given by the equation

$$t = \frac{\theta - \theta_0}{\theta_1 - \theta_0} = \frac{\arctan \frac{y}{x} - \theta_0}{\theta_1 - \theta_0}$$

and the concentric circles would be the lines of flow.

By the mapping

$$w = e^z,$$

discussed in chapter I, §3, the rectangular subdivision, which represents the temperature condition of the parallel strips, is transformed into the rectangular subdivision by the concentric rings and pencil of rays. Thus the temperature condition of the concentric rings is a con-

formal representation of the parallel strips.

5. Heat applied in one point and drawn off in another. If we have given two non-concentric circles

$$\log \frac{p}{q} = \alpha_0 , \quad \log \frac{p}{q} = \alpha_1$$

and give them, respectively, the temperature 0° and 1°, the isothermes will be

$$\log \frac{p}{q} = \alpha$$

These circles are the conformal representation of the circles

$$\log r = \alpha$$

by means of the function

$$w = \frac{1}{z} ,$$

which is discussed in chapter I, § 4. The lines of flow will be the orthogonal system of circles through the

two points,

$$\phi - x - y$$

The temperature of the isotherms is given by the equation

$$t = \frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0} = \frac{\frac{1}{2}\log\frac{(x-a)^2+(y-b)^2}{(x-a_1)^2+(y-b_1)^2} - \alpha_0}{\alpha_1 - \alpha_0}$$

In this form $\Delta u = v$ is satisfied. Hence t may be taken as the u part of a complex function, the v part is the orthogonal system

$$\phi - x - y$$

This temperature condition is represented in Fig. 3. The non-concentric circles

$$\log\frac{\rho}{\rho'} = \alpha$$

are the isotherms. The lines of flow are the pencil of circles through the points 0 and $\frac{1}{a+ib}$. In other words,

one point is a constant source of heat, at the other the heat is drawn off.

This temperature condition is a conformal mapping of the condition discussed in § 4, by means of the function

$$w = \frac{1}{z}$$

In other words, these two isothermic conditions are connected by the relation

$$w = \frac{1}{z}$$

where z is a complex variable.

As before the lines of streaming and the isotherms may be interchanged.

6. Other problems. Difficulties. The foregoing paragraphs show the general method of the applications of the complex variable to heat. The applications to electricity are similar, and,

in fact, more practical to obtain. We shall, therefore, take that in the next chapter more in detail, making this somewhat briefer, and keeping in mind that the heat condition is similar to that of the potential function.

The difficulties we come across in applying this theory to heat lie in the fact that heat radiates out in all directions and we cannot in practice limit the flow to space of two dimensions. We may, however, approximately reach the desired conditions (a) by maintaining the medium about the surface at constant temperature or by considering a plane

the cross section of a solid, in which the flow of heat takes place, © or by using a plane surface whose conductibility is considerably above that of the surrounding region.

Chapter III

Applications to Electricity

1. Introduction. In § 5 of the introduction, we defined equipotential lines and lines of streaming. In electricity the equipotential lines correspond to the isotherms and the potential function to the function which represents the temperature condition.

The simplest case is that in which the equipotential lines are concentric circles. The potential function is given by the equation.

(1) $$v = \frac{\log \frac{r}{r_0}}{\log \frac{r_1}{r_0}} = c \log r \ldots$$

[Compare equation (1) § 4, chapter II] This is called the logarithmic potential. In this field of logarithmic potential belong problems in stationary electric streaming in an infinite or, in some cases, bounded plane plate. In this plate there should be a constant streaming in and out; but the number of positive and negative electrodes need not be the same. The condition that the streaming be constant is that the potential function v satisfy Laplaces equation. Hence the potential function v may be taken as the v part of a complex function. The function of flow will be the conjugate function u which can be determined,

except as to a constant, when v is known.

2. Electrical streaming from one point out into an infinite region. For this case Kirchoff* has shown both theoretically and experimentally that the potential function is

$$V = -c \log r \tag{1}$$

[Compare equation (1) §1.]

where

$$c = \frac{E}{2\pi k \delta}$$

E is the electro-motive force, k the specific conductive capacity, and δ the arbitrarily small thickness of the plate. If in (1) we substitute

$$r = \sqrt{(x-a)^2 + (y-b)^2}$$

* Kirchoff: Ueber den Durchgang eines elektrischen Stromes durch eine Ebene, insbesondere durch eine kreisförmige. - Poggendorf's annalen, Vol. 64, 1845 and Vol. 67, 1846

the potential v satisfies Laplace's equation and hence may be taken as the imaginary part of the complex function. The practical meaning of the potential function v is the strength of the electro-motive force in the different points of the plate.

The curves

$$\log r = \log \sqrt{(x-a)^2+(y-b)^2} = \alpha,$$

which are concentric circles, represent the equipotential lines. The curves

$$\varphi = \arctan \frac{y-b}{x-a} = \gamma,$$

which are the orthogonal pencil of rays, represent the lines of streaming. The geometric representation of these curves is given by the mapping of the function

$$w = e^z$$

where z is a complex variable. This mapping is discussed on page 23. Fig. 2 shows the lines of flow and of equipotential.

3. Case when there is one positive and one negative electrode. By the mapping $w = \frac{1}{z}$ the stream lines from the point $a+ib$ to ∞,

$$\varphi = x$$

become the stream lines

$$\varphi - x = y \tag{11}$$

through the points $\frac{1}{a+ib}$ and 0. [See Fig. 3.] This mapping is discussed on page 28. If the point $a+ib$ is the positive electrode and ∞ the negative electrode (the case in §2), then by this transformation $\frac{1}{a+ib}$ becomes the positive electrode and 0 the negative.

We have, then, the representation of the stream lines when electricity enters at one point and leaves at another point. The orthogonal system of curves

(2.) $$\log \frac{r}{r_1} = a$$

represent the equipotential lines. In chapter I, § 4, we showed that (1.) and (2.) written in Cartesian co-ordinates satisfied Laplace's equation. Hence the curves which they represent may be taken as the lines of flow and of equipotential.

We might have approached this case in the opposite way by inquiring what would be the curves when electricity is applied at one point and withdrawn

ff at another. We would then consider which transformation of the simple case, where electricity enters at the point $a+ib$ and leaves in an infinite region, will give one point corresponding to $a+ib$ and one corresponding to ∞. Such a transformation is

$$w = \frac{1}{z}$$

This latter way of applying conformal representation is the more difficult and, for the more complicated problems, requires an intimate knowledge of isogonal transformations. It would however, seem to be of more practical use to employ conformal representation for the discovery of the lines of

flow and the equipotential lines when electricity is applied in different places, without having to resort to experiment. Both methods have their value. The first brings out interesting and perhaps unknown facts, showing different representations and how electricity should be applied to obtain the given lines of flow and equipotential lines.

4. Two positive electrodes, negative electrode at infinity. We shall now consider what will be the character of the curves when electricity enters in two points and leaves in an infinite region. In the mapping

$$w = \sqrt{z}$$

point
to every finite of the z-plane, there
correspond two in the w-plane.
When z equals ∞, w also equals ∞.
If then we map the stream lines

$$\varphi = t$$

and the equipotential lines

$$\log r = a$$

upon the w-plane by means of the
function

$$w = \sqrt{z},$$

we shall obtain the geometric repre-
sentation of the stream and equipoten-
tial lines when electricity enters the
plane plate in two points with equal
strength. This mapping is discussed
in chapter 1, § 6. The curves are shown
in Fig. 6. The stream lines are the

equilateral hyperbolas

$$\theta + \theta_1 = y,$$

the equipotential lines are the confocal lemniscates

$$\rho \cdot \gamma = a.$$

5. Case when the positive and negative electrodes are lines. By the mapping

$$w = z^2,$$

the parallels to the u and v axes become equilateral hyperbolas. The line

$$u = 0$$

corresponds to the lines

$$x = 0, \quad y = 0$$

The line

$$v = 0$$

corresponds to the lines bisecting the angles between the x and y axes. This

mapping is shown in Fig. 4.

If we let the line

$$v = b$$

be the positive electrode, the line

$$v = -b$$

the negative electrode, and if our plate is bounded on two sides by these lines, then the stream lines are given by

$$u = c.$$

The equipotential lines are given by

$$v = c_1.$$

By the transformation

$$w = z^2,$$

the lines

$$v = b \quad , \quad v = -b$$

become the equilateral hyperbolas

$$2xy = b \quad , \quad 2xy = -b.$$

These equations, then, represent the electrodes. In the figure they are marked by the heavy lines. Since z is a double valued function of w, we should expect two positive and two negative electrodes.

If $b = \infty$, the electrodes will be straight lines. Since for most problems in physics ∞ is only a relative number, it would be possible to obtain, approximately correct curves by sending a current through a moderately large plate, as shown in the figure, when the heavy dotted lines represent the electrode.

6. The case when the positive electrode is a line, the negative electrode is at infinity.

If we let the x-axis be a positive electrode, and the negative electrode be at ∞, the stream lines will be

$$x = c$$

and the equipotential lines will be

$$y = c_1 .$$

On transforming this potential condition to the w-plane by means of the function $w = z^2$, the stream lines become the parabolas

$$(1) \qquad u + c^2 = \left(\frac{v}{2c}\right)^2 ,$$

and the equipotential lines become the parabolas

$$(2) \qquad c^2 - u = \left(\frac{v}{2c}\right)^2$$

The x-axis corresponds to that part of the u-axis which extends from $v = 0$ to $v = \infty$. The y-axis corresponds to

the negative half of the u axis. In the w-plane, then, the positive electrode is the positive half of the w-axis. The negative electrode is at ∞. The representation of this electrical condition is shown in Fig. 5. The heavy line represents the positive electrode. This mapping is discussed on page 39.

We might have taken the y-axis in z-plane as positive electrode. The stream lines would then have been interchanged with the equipotential lines, and the negative half of u-axis would have been the positive electrode.

If we had taken the positive electrode at

$x = 0$

the negative at

$$x = -\alpha$$

by the transformation, both electrodes would have been at α. The positive would have been the upper half, the negative electrode the lower half of the parabola

$$u = \alpha,$$

which is a line since from (2.)

$$\alpha^2 - u = \left(\frac{v}{2\alpha}\right)^2 = 0$$

Hence

$$u = \alpha$$

In the figure the heavy dotted lines represent the electrodes for this case.

7. *When electricity enters at two points and leaves at two points.* We shall now

consider what are the lines of flow and of equipotential when electricity enters at two points and leaves at two points. We have the geometric representation and the analytical expression for the case when electricity enters at one point and leaves at one point. In this case the stream lines are the pencil of circles

$$\varphi - x = y$$

through the points $a+ib$ and a_1+ib_1, the points $a+ib$ and a_1+ib_1 being the electrodes. By the mapping

(1) $$w = \sqrt{z},$$

two points of the w-plane correspond to one of the z-plane. This mapping is discussed in chapter 1, § 7. If then

by means of (1), we map the pencil of circles

$$\varphi - \lambda - \gamma$$

and its orthogonal system

$$\log \frac{r}{q} = c$$

upon the w-plane, to the two points $a+ib$ and a_1+ib_1 there correspond the points $\pm\sqrt{a+ib}$ and $\pm\sqrt{a_1+ib_1}$. Since $a+ib$ and a_1+ib_1 are the electrodes, by this transformation, we obtain the representation of the lines of flow and of equipotential, when we have the two positive electrodes $\pm\sqrt{a+ib}$ and the two negative electrodes $\pm\sqrt{a_1+ib_1}$. This mapping is shown in Figs. 7 and 8. In Fig. 7, the electrodes are corners of a parallelogram while in Fig. 8 they

are the corners of a square. A and A' are the positive electrodes, B and B' are the negative electrodes. The arrows indicate the directions of flow. Fig. 9 represents the special case where the point $a, +ib,$ is 0 and we have the two negative electrodes at the origin or one negative electrode of double strength at the origin. A and A' indicate the positive electrodes; B, the the negative electrode.

8. Physical interpretation of Fig. 10. In mapping the z-plane upon the w-plane by means of the functions

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

the stream lines

$$\varphi = \chi$$

correspond to a system of equilateral hyperbolas about the points +1 and -1. The equipotential lines

$$r = c$$

correspond to a system of confocal ellipses about the points ±1. The line from +1 to -1 represents the positive electrode, the negative electrode is at infinity. In Fig. 10 this electrical condition is shown. The heavy line is the positive electrode. The arrows on the hyperbolas indicate the direction of flow.

If we interchange the function of flow with that of potential, the ellipses become the lines of streaming; the hyperbolas, the equipotential lines. The positive electrode is, then, a

portion of the u-axis, that part which extends in a positive direction from +1 to ∞. The negative electrode is that part of the u-axis which extends from -1 to ∞ in a negative direction. The heavy crossed lines indicate these electrodes.

In the chapter on Conjugate Functions, §192, Maxwell discusses this electrical condition. However, he bases his analytical calculations on the properties of conjugate functions of two real variables.* This is equivalent to basing it upon functions of a complex variable since in

$$w = u + iv$$

u and v are conjugate functions.

* "Electricity and Magnetism", Vol. 1. - Clerk Maxwell.

9. Case when there are two positive electrodes, one negative electrode half way between them and one at infinity. We shall now discuss the physical meaning of the mapping

$$w = z + \sqrt{z^2 - 1}$$

which was considered in § 9, Chapter I. and illustrated in Fig. 11. By this transformation the stream lines

$$\theta = \chi$$

become the stream lines

$$\varphi - \chi = \theta = \chi.$$

The equipotential lines

$$r = c$$

become the equipotential lines

$$\frac{p \cdot q}{2r} = c.$$

When written in the real form in

Cartesian co-ordinates, (1) and (2) satisfy Laplace's equation and therefore may be considered as representing the lines of flow and of equipotential of a new electric potential condition.

By the transformation, we obtain two positive electrodes and two negative electrodes, one of the negative electrodes being at ∞, the other half way between the two positive electrodes. In the figure A and A' indicate the positive poles, B the finite negative pole. The arrows indicate the direction of flow.

If we had taken the curves

$$\frac{\mathrm{P}\cdot\Psi}{2\pi} = c$$

i.e. stream lines and the curves

$$\varphi + \chi - \theta = \chi$$

as equipotential lines, then we would have had two positive line electrodes, one from A' to ∞, the other from A to B. The two negative electrodes would be, one the portion of y-axis from B to A', the other the portion from ∞ to A. The heavy lines in the figure indicate the positive electrodes; the crossed lines, the negative electrodes.

Chapter IV

Applications to Hydrodynamics

1. Introduction. When the motion of a liquid takes place in a series of planes parallel to the xy plane and is the same in each plane, the whole motion is known, if we know that in the plane $z=0$ where x, y, and z are the rectangular co-ordinates of space. To such a movement of liquids the complex variable is applicable.

In the case of fluid motion where the motion is virotational, the stream-function u must satisfy the equa-

tion of continuity*

$$\Delta u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

The velocity-potential v must satisfy

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$$

The curves u - const. are the stream lines, the curves v - const. the equipotential lines. The functions u and v are connected by the relations

$$\frac{du}{dx} = \frac{dv}{dy}, \quad \frac{du}{dy} = -\frac{dv}{dx}$$

These are the conditions that $u + iv$ must satisfy in order that it be a function of the complex variable $x+iy$. Hence any assumption that

$$u + iv = f(x+iy)$$

gives a possible case of irrotational motion.

*"Hydrodynamics" — Horace Lamb. pub. 1895. — § 59 page 68

It is evident of course that the curves

$$u = \text{const.}, \quad v = \text{const.}$$

form two orthogonal systems.

2. Applications of the function $w = Az^n$.

In the foregoing paragraph, we have shown that the functions of a complex variable may be applied to the determination of cases of fluid motion in two dimensional space. We shall now explain some of these applications.

First, we assume

$$w = Az^n,$$

where A is any real constant. We not the following cases.

I If $n = 1$, the stream lines

$$u = \text{const.}$$

are a system of straight lines

parallel to x-axis. The equipotential lines

$$v = \text{const.}$$

are parallels to y-axis. In this case the corresponding figures in the w and z planes are similar.

If $n = 2$ the curves

$$u = \text{const.}$$

are a system of equilateral hyperbolas, having the axes of co-ordinates as their principal axes. The curves

$$v = \text{const.}$$

are a similar system, having the co-ordinate axes as asymptotes. This mapping is shown in Fig. 4. We may take the axes of x and y as fixed boundaries. If we consider the positive parts of the x and

y axes as fixed boundaries, we have the case of a fluid in motion in the angle between two perpendicular walls.

3. Application of the function $w = z^2$. The mapping of the function

$$w = z^2$$

was explained in § 3, Chapter I and is shown in Fig. 2. If we take the pencil of rays

$$\varphi = y$$

as the stream lines, we get the case of a (two dimensional) source at the origin.

If the circles

$$r = c$$

be taken as stream lines, we get a cyclic motion, the circulation in any circuit embracing the origin

being 2π.*

4. The case of a source and a sink. By the mapping

$$w = \frac{1}{z}$$

the curves

$$\varphi = \chi$$

$$r = c$$

of the z-plane correspond to the curves

$$\varphi - \chi = \gamma$$

$$\frac{r}{\rho} = c$$

of the w-plane. This mapping is discussed in § 4, chapter II and illustrated in Fig. 3.

If the circles

$$\varphi - \chi = \gamma$$

* Lamb's "Hydrodynamics" Art 98.

be taken as the stream lines, we have the case of a source and sink of equal intensities, situated the one at the origin the other at the point $\frac{1}{a+ib}$

If the circles

$$\frac{r}{\rho} = c$$

be taken as the stream lines, we get a case of cyclic motion. The circulation in any circuit embracing the first (only) of the above points, 0 and $\frac{1}{a+ib}$, is 2π. That in a circuit embracing the of these points is -2π. On the other hand if the circuit embraces both the circulation is zero.

5. Application of the function $w = \frac{1}{2}(z + \frac{1}{z})$.

By the mapping

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

the curves

$$\eta = Y$$

and the curves

$$\xi = c$$

of the z-plane correspond respectively to the curves

$$\frac{P-Q}{2} = \cos y$$

$$\frac{P+Q}{2} = \frac{1}{2}\left(c + \frac{1}{c}\right)$$

of the w-plane. This mapping is shown in Fig. 10.

If the hyperbolas be taken as the stream-lines, the portions of the axis of x which lie outside the points ± 1 may be taken as rigid boundaries. We obtain in this way the case of a liquid flowing from one side to the other of a

thin plane partition, through an aperture of breadth 2.

If the ellipses be taken as the stream-lines, we get the case of a liquid circulating round an elliptic cylinder.

6. Application of the function $z = w + e^w$.

This mapping was not discussed in chapter I, therefore we shall consider it now. Given that

$$z = x + iy = w + e^w = u + iv + e^u \cdot e^{iy}$$

Then it follows from Euler's relations that

$$x + iy = u + iv + e^u(\cos v + i \sin v)$$

Equating real and imaginary parts, we have

$$x = u + e^u \cos v$$

$$y = v + e^u \sin v$$

The stream line $v = 0$ coincides with the axis of x. Again the portion of the

the line

$$y = \pi$$

between $x = \infty$ and $x = -1$, considered as a line bent back on itself, forms the stream line $v = \pi$. That is to say as u decreases from $+\infty$ through 0 to $-\infty$, x increases from $-\infty$ to -1 and then decreases to $-\infty$ again. Similarly for the stream line $v = -\pi$. The equipotential lines or u curves are the orthogonal system*. The lines of flow are shown in the following figure.

This function, when written in the real form in Cartesian co-ordinates, satisfies Laplace's equation. The mapping

* Compare, §212, "Electricity and Magnetism." – Maxwell.

is therefore conformal and may be applied to the movement of liquids.

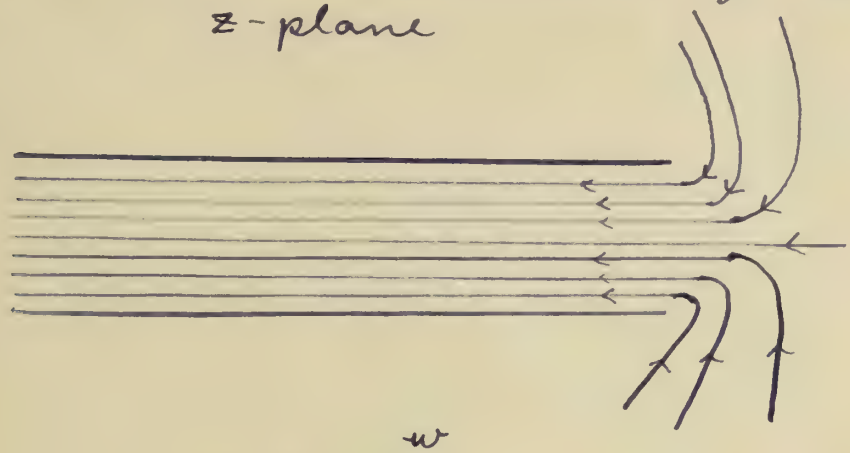


Fig. 13

$$z = w + e^{w}$$

The above mapping represents the movement of a liquid from an open space into a canal bounded by two thin parallel walls. If we change the sign of w, the direction of the flow will be reversed and we would have a liquid flowing from a canal into an open space.

Applications of the motion of liquids to other mappings explained in chapter I are similar to the flow of electricity,

we therefore shall not discuss them further.

Chapter V

Experiments.

1. Introduction. To show that isogonal transformation could be used, practically, to determine the character of the lines of flow and of equipotential, we have performed a few experiments in the physics laboratory. The experiments had to do with the flow of electricity in a plane plate.

2. Method of experiment. We have the fact that if either the u curves or v curves are known the others may be determined except as to

constant. Also we know that if either the lines of flow or of equipotential are known the other curves may be obtained, since the two systems are orthogonal by definition. Hence, if we should determine the u or the v curves by experiment, the others could be calculated. We accordingly determined the equipotential lines of the flow of electricity through a plane plate, when the electricity was applied in different places. Most of our theory applied to infinite planes; for this reason, we made use of a moderately large tin plate, 1½ by 2 ft. in dimensions. By use of a plate of this size the edge had a decreased effect upon the figures obtained.

Wires carrying a current were soldered to the points on the plate at which we wished the current to enter and leave. A continuous current, then, flowed through the plate from the positive to the negative terminals.

Points having the same potential were determined by the "zero" method. These points were connected by a smooth curve, which curve represented an equipotential line.

The "zero" method of determining equipotential points me be described as follows. Two points of the plate are connected through a galvanometer. Where there is a difference of potential in the two points, it will cause a

deflection in the galvanometer. One point is then held fixed, the other varied until one is found which produces no deflection of the galvanometer. This means that the two points have the same potential. In our experiments, we determined a sufficiently large number of equipotential points to show the character of the curve.

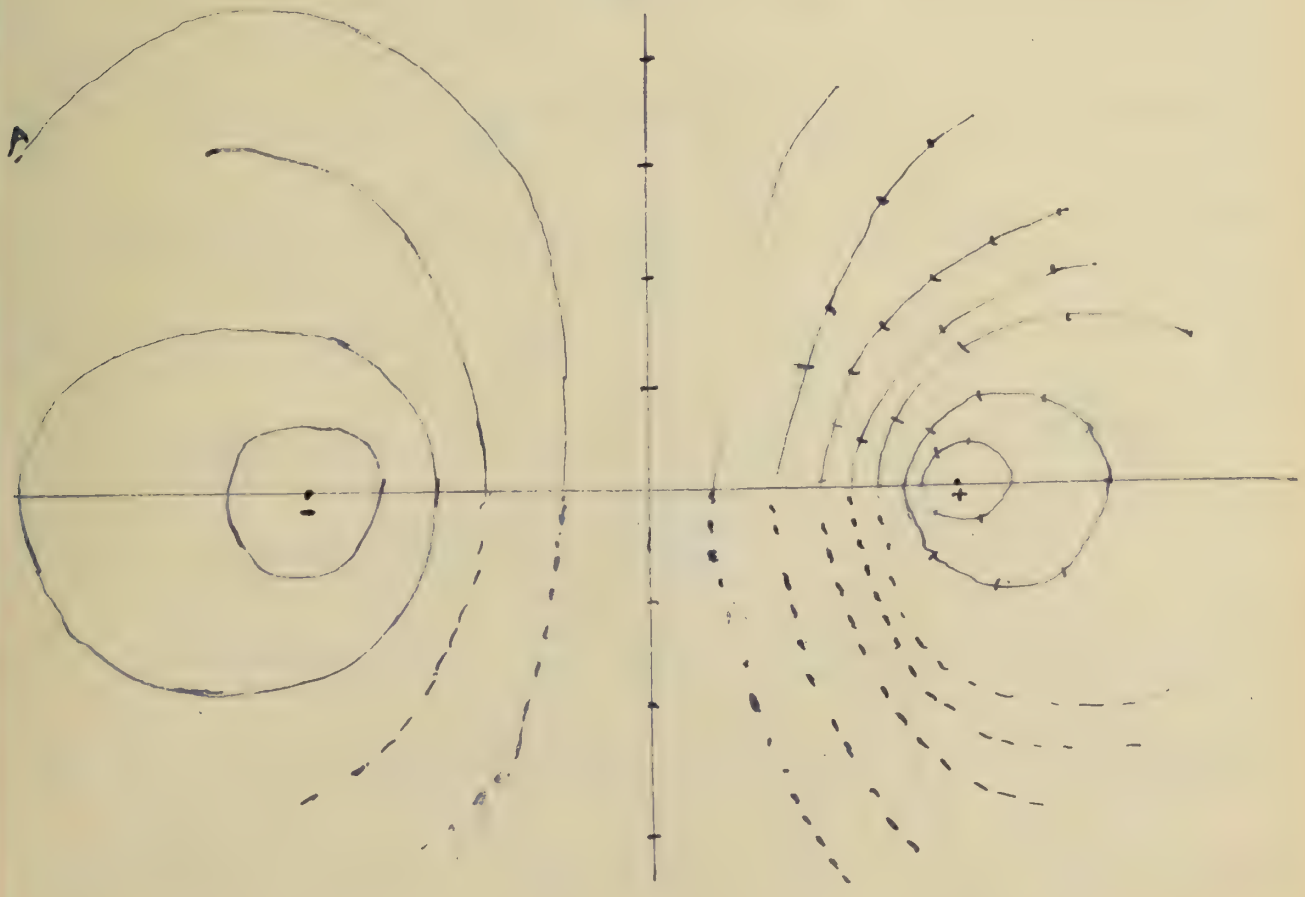
3. Sources of error. There were a number of sources of error which must be taken into account, when comparing the figures obtained by experiment with those obtained by analytical calculation. The plate used was of sheet iron rolled in tin, hence the tin was not necessarily equally

distributed over the entire surface, and as a consequence the resistance would not be the same throughout. Another source of error lay in the effect of the edges of the plate on the flow of the current, even with the moderately large plate employed. The terminals of the battery could not be soldered to the plate in one point, so that the current sometimes entered the plate in a place a half-centimeter in length, and the diameter of the wire in width. Herein, then, lay a source of error. Also the terminals through the galvanometer made contact with the plate in more than one point.

When these sources of error are taken into account, the results of the experiments show that our analytical calculations are correct.

4. Results of the experiments. The following figure represents the equipotential lines obtained in the first experiment.

Fig. 14



The positive pole of one storage cell was connected with the plate at one point represented by + in the figure. The negative pole was connected at a point twenty-six centimeters distant from the positive pole. It is indicated by − in the figure. Fig. 14. is reduced to one-third the size of the figure obtained by experiment. The smooth curves are the equipotential lines drawn through the equipotential points as determined by the experiment. The dotted curves are filled in from the upper plane.

Comparison of this figure with that of Fig. 3, which illustrates the character of the lines of equipotential obtained

by analytical calculation, shows them to be similar. The irregularity of some of the curves of Fig. 14 is due to the sources of error already explained. The upper part of the curve A illustrates the effect of the edge of the plate.

Fig. 15 represents the equipotential lines when electricity enters at two diagonally opposite corners of a square and leaves in the other two corners. We determined by experiment the equipotential lines of but one quadrant, and drew the others in, since in each quadrant the curves would be similar. The dotted lines represent the curves drawn in. These curves should be compared with those

of Fig. 8. It will be seen at once that the equipotential lines obtained by experiment belong to the same system as those obtained by analytical calculation.

Fig. 15

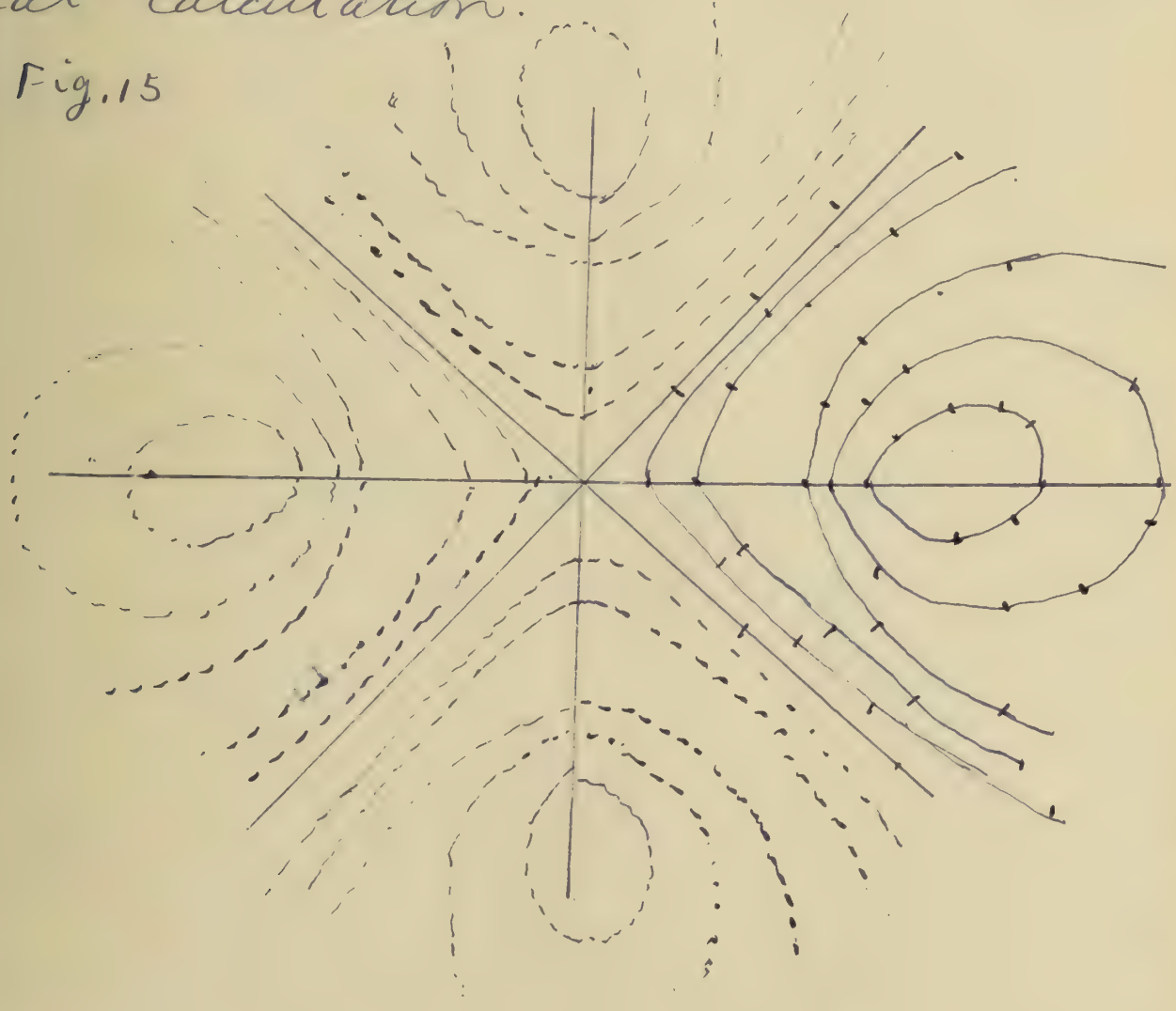
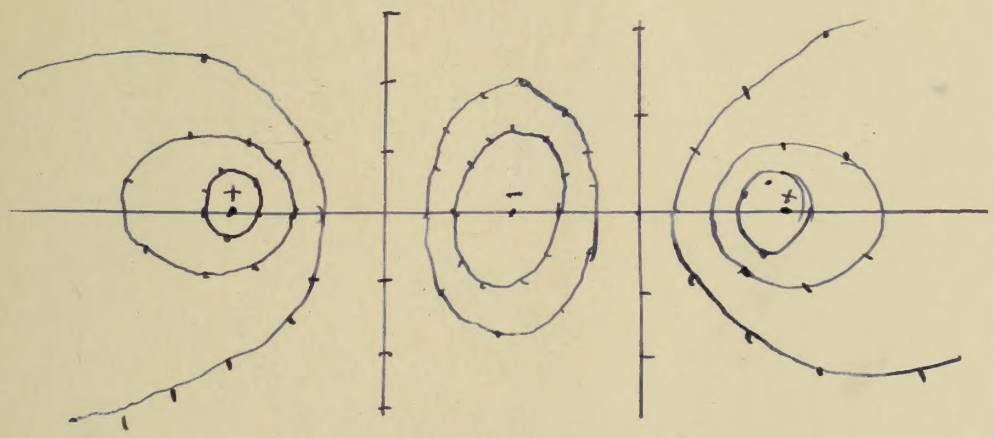


Fig 16, shows the equipotential lines when electricity enters at two points

and leaves at a point midway between them. This figure should be compared with Fig. 2. which represents the equipotential curves when electricity enters the plane in this manner.



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